Newtonian spherical gravitational collapse

1980 J. Phys. A: Math. Gen. 13 3097

(http://iopscience.iop.org/0305-4470/13/9/035)

J. Phys. A: Math. Gen. 13 (1980) 3097–3104. Printed in Great Britain

Newtonian spherical gravitational collapse†

E N Glass

Department of Physics, University of Windsor, Windsor, Ontario, Canada N9B 3P4

Received 19 November 1979, in final form 10 March 1980

Abstract. The spherical collapse of gas clouds is reduced to quadrature by specifying the mass function $m(r, t)$. Exact solutions for both shearing and shear-free flow are constructed admitting two-parameter equations of state. A generalisation of Ritter's theorem is proved which allows the shear-free collapse of index three polytropes.

1. Introduction

The spherical collapse of gas clouds is relevant to the understanding of star formation. Numerical studies (see Woodward's (1978) review for references) generally show initial isothermal collapse of an optically thin outer region onto a central region which evolves sufficiently high opacity to form a core. The outer region then collapses onto the core.

The stability of such collapse against fragmentation remains an important topic to be understood. Hunter's (1967) studies of uniform clouds showed instability to fragmentation. Since numerical work indicates that highly non-uniform collapse best models actual physical conditions, it is necessary to have exact non-uniform collapse solutions in order to study their stability. This work reduces the collapse problem to quadrature and presents a method for generating exact collapse solutions which satisfy all physical boundary conditions. In addition, the solutions generated here are free of singularities, while the known exact similarity solutions, such as those of Hunter (1977) and Cheng (1978), have singularities in their domain.

Section 2 presents the dynamical equations for spherical collapse, and shear-free flow is considered in § 3. A generalised version of Ritter's theorem is given which allows spherical shear-free collapse for $n = 3$ polytropes. In § 4 boundary conditions are given for all the physical variables. Exact solutions for shear-free collapse are constructed in § 5. The time development for collapse is discussed in § 6, and examples of collapse with shear are given in § 7.

2. Dynamics

The spherical collapse of an inviscid, non-heat conducting gas obeys the momentum balance equation

$$-\frac{\partial p}{\partial r} = \rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{Gm}{r^2}\right). \tag{1}$$

† Supported in part by the Natural Sciences and Engineering Research Council of Canada.

0305-4470/80/093097+08$01.50

The mass function $m(r, t)$ must satisfy

$$\partial m/\partial r = 4\pi r^2 \rho, \tag{2}$$

$$\partial m/\partial t = -4\pi r^2 \rho v, \tag{3}$$

where the continuity equation is the integrability condition for equations (2) and (3). It is usual to complete the system by specifying a barotropic equation of state $p = p(\rho)$, thus providing four equations for the unknown p, ρ, v, and m. However, a collapse may have not a single equation of state, but rather a two-parameter family of them. This is reasonable for ideal systems which model actual physical conditions with viscosity, heat flow, and time varying opacity.

The approach taken here is to specify $m(r, t)$ explicitly. This reduces the collapse problem to a quadrature since equation (2) yields ρ, equations (2) and (3) yield v, and then equation (1) requires a single integration. The form of m is constrained by the boundary conditions.

3. Shear-free flow

The velocity of spherical matter flow is given in the Eulerian frame by

$$\boldsymbol{v} = v(r, t)\boldsymbol{e}_r,$$

with acceleration

$$a\boldsymbol{e}_r = (\partial v/\partial t + v\,\partial v/\partial r)\boldsymbol{e}_r.$$

The rate of expansion of the flow is

$$\Theta := \boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v), \tag{4}$$

and the rate of shear is given by†

$$\sigma_a{}^b = (\partial v/\partial r - \tfrac{1}{3}\Theta)\delta_a^{(r)}\delta_{(r)}^b + (v/r - \tfrac{1}{3}\Theta)(\delta_a^{(\theta)}\delta_{(\theta)}^b + \delta_a^{(\phi)}\delta_{(\phi)}^b). \tag{5}$$

The collapse will be shear-free if, and only if,

$$v = rH(t), \tag{6}$$

for any $H(t)$. Shear-free flow is very restrictive and, in general, can represent only a portion of the more realistic collapse models (see McVittie (1956) for shear-free solutions with heat flow). It has been shown by Mansouri (1977) and Glass (1979) that a general relativistic perfect fluid in spherical shear-free collapse can never admit a barotropic equation of state $p = p(\rho)$. The Newtonian case‡ is less restrictive and allows a generalised version of Ritter's theorem (Chandrasekhar 1958, p 48) which is given following the lemma.

† Indices within parentheses indicate coordinate values with respect to standard r, θ, ϕ spherical coordinates.
‡ The statement in Glass (1979) that the general relativistic proof holds in the Newtonian case is herein shown to be false.

Lemma. The general form of the mass function for spherical shear-free collapse is given by

$$m = F[r/\lambda(t)], \tag{7}$$

where F and $\lambda(t)$ are arbitrary functions.

Proof. The lemma follows upon requiring $v = rH(t)$ and then noting that equation (7) is the complete solution of

$$\partial m/\partial t + rH(t)\partial m/\partial r = 0,$$

with $H(t) = \dot{\lambda}/\lambda$, and where overdots henceforth symbolise $\mathrm{d}/\mathrm{d}t$ of time functions.

Theorem. An inviscid, non-heat conducting gas sphere in shear-free collapse (or expansion) can have a polytropic equation of state which is either index three, or pressure-free uniform density.

Proof. Assume shear-free flow with velocity $v = (\dot{\lambda}/\lambda)r$, and equation of state $p = A\rho^{\gamma}$, A and γ constant. From equation (7) of the lemma

$$m = F(\phi), \qquad \phi := \lambda^{-1}r.$$

The polytropic equation of state and equation (2) for the density are substituted into equation (1), yielding

$$A\gamma(4\pi)^{1-\gamma}\left(\phi\frac{\mathrm{d}F}{\mathrm{d}\phi}\right)^{\gamma-2} r^{-3(\gamma-2)-2}\left(\phi^2\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2} - 2\phi\frac{\mathrm{d}F}{\mathrm{d}\phi}\right) + r^3(\ddot{\lambda}/\lambda) + GF = 0. \tag{8}$$

In order that equation (8) have a well defined solution $F(\phi)$ two conditions must hold:

(a) $\ddot{\lambda} = -\frac{1}{2}k_1\lambda^{-2}$, k_1 constant;

(b) $-3(\gamma-2)-2 = 0$, when $F \neq \text{constant}\ (\phi^3)$.

Condition (b) implies

$$\gamma = \tfrac{4}{3} \qquad \text{or} \qquad n = 3 \tag{9}$$

($\gamma = 1 + 1/n$ where n is the polytropic index). Equation (8) becomes

$$\phi^2\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2} - 2\phi\frac{\mathrm{d}F}{\mathrm{d}\phi} + k_3\left(\phi\frac{\mathrm{d}F}{\mathrm{d}\phi}\right)^{2/3}(GF - \tfrac{1}{2}k_1\phi^3) = 0, \qquad k_3 := \tfrac{3}{4}A^{-1}(4\pi)^{1/3}. \tag{10}$$

Equation (10) admits three cases.

(i) $F = (k_1/2G)\phi^3$. This solution implies $\rho = \rho(t)$ with equation of state $p = 0$. The time dependence is given by solutions of $\dot{\lambda}^2 = k_1\lambda^{-1} + k_2$.

(ii) Ritter's case, $k_1 = 0$ with $n = 3$. The acceleration $a = (\ddot{\lambda}/\lambda)r$ is zero in this case.

(iii) $k_1 \neq 0$ with $n = 3$. A numerical solution exists for this case.

4. Boundary conditions

Density: At the origin, the central density $\rho(0, t)$ must be a positive function, bounded as $t \to \infty$ for either collapse to a static state or oscillations about an equilibrium configuration. At the outer boundary, ρ must be zero or have some small positive value.

Pressure: The outer boundary of the collapsing gas is given by $p(r_b, t) = 0$. The central pressure $p(0, t)$ must be positive and bounded as $t \to \infty$. In addition $\partial p/\partial r = O(r)$ as $r \to 0$.

Mass: In order that the density be finite (non-zero) at the origin, $m = O(r^3)$ as $r \to 0$. The total mass of the system is conserved and so $M := m(r_b, t)$ must obey $\dot{M} = 0$.

Velocity: Equation (3) implies that $v = O(r)$ as $r \to 0$ for consistency with the mass and density conditions. At the outer boundary, $v(r_b, t)$ must be finite, less than c, and satisfy

$$\mathrm{d}r_b/\mathrm{d}t = v(r_b, t).$$

5. Shear-free collapse examples

5.1.

Following the functional form given in equation (7) of the lemma, and with the central behaviour of m restricted to $O(r^3)$, we choose

$$m = 4\pi m_0 \lambda^{-3} r^3 (1 + \lambda^{-2} r^2)^{-3/2}, \tag{11}$$

where m_0 is a positive constant and $\lambda(t)$ is a function to be determined. Equations (2) and (3) yield

$$\rho = 3m_0 \lambda^{-3}(1 + \lambda^{-2} r^2)^{-5/2}, \tag{12}$$

$$v = (\dot{\lambda}/\lambda) r. \tag{13}$$

Substituting (12) and (13) into equation (1) and integrating provides the pressure:

$$p = m_0 \ddot{\lambda} \lambda^{-2}(1 + \lambda^{-2} r^2)^{-3/2} + (2\pi G m_0^2)\lambda^{-4}(1 + \lambda^{-2} r^2)^{-3} + h(t), \tag{14}$$

where $h(t)$ is a function of integration. Since $h(t)$ can be freely chosen, any function $r_b = r_b(t)$ can be made to satisfy $p(r_b, t) = 0$. Hence we examine the mass function to determine the boundary. $m(r_b, t)$ is constant for†

$$r_b = \lambda(t), \tag{15}$$

with the result

$$M := m(r_b, t) = \sqrt{2}\pi m_0.$$

The boundary location (15) determines $h(t)$ in equation (14), and so the pressure becomes

$$p = m_0 \ddot{\lambda} \lambda^{-2}[(1 + \lambda^{-2} r^2)^{-3/2} - 2^{-3/2}] + (2\pi G m_0^2)\lambda^{-4}[(1 + \lambda^{-2} r^2)^{-3} - 2^{-3}]. \tag{16}$$

In the limit $\lambda(t) \to \lambda_0$ constant, the static solution is the $n = 5$ Lane–Emden polytrope with equation of state

$$p = k_1 \rho^{6/5} - k_2.$$

5.2.

We make the choice

$$m = 4\pi m_0[\sin(\lambda^{-1} r) - \lambda^{-1} r \cos(\lambda^{-1} r)]. \tag{17}$$

† Since $m = m(r/\lambda)$, any choice $r_b = \text{constant}(\lambda)$ will make $m(r_b, t)$ constant. Equation (12) does not allow a choice such that $\rho_b = 0$ so, for simplicity only, we choose $r_b = \lambda$.

Equations (1), (2) and (3) yield

$$\rho = m_0\lambda^{-3}[\sin(\lambda^{-1}r)]/\lambda^{-1}r, \tag{18}$$

$$v = (\dot{\lambda}/\lambda)r, \tag{19}$$

$$p = m_0\left(\frac{\ddot{\lambda}}{\lambda^2}\right)\cos(\lambda^{-1}r) + \frac{2\pi G m_0^2}{\lambda^4}\left(\frac{\sin(\lambda^{-1}r)}{\lambda^{-1}r}\right)^2 + h(t). \tag{20}$$

The mass function is constant, and $\rho_b = 0$, for

$$r_b = \pi\lambda(t), \tag{21}$$

which yields $M = 4\pi^2 m_0$ and determines $h(t)$ such that

$$p = m_0\left(\frac{\ddot{\lambda}}{\lambda^2}\right)[\cos(\lambda^{-1}r) + 1] + \frac{2\pi G m_0^2}{\lambda^4}\left(\frac{\sin(\lambda^{-1}r)}{\lambda^{-1}r}\right)^2. \tag{22}$$

The static limit $\lambda(t) \to \lambda_0$ is the $n = 1$ Lane–Emden polytrope with equation of state

$$p = (2\pi G\lambda_0^2)\rho^2.$$

6. Time development

For collapse to static equilibrium, a generic time profile is sketched in figure 1. At $t = 0$, the system is gravitationally bound and starts collapse with zero velocity and inward acceleration. In order for the velocity to reach zero again, it must have an extremum at which time the acceleration is zero. Static equilibrium is achieved only when the velocity and acceleration are simultaneously zero, and so the acceleration must have an outward maximum before its second zero. It is clear that the outward maximum of acceleration can be produced only by having the pressure overshoot its final equilibrium value.

More complicated time profiles are easily imagined: the acceleration and velocity curves may cross any number of times before achieving a simultaneous zero, or the system may settle down to steady oscillations about some equilibrium configuration with acceleration and velocity never finding a simultaneous zero. Two extreme cases

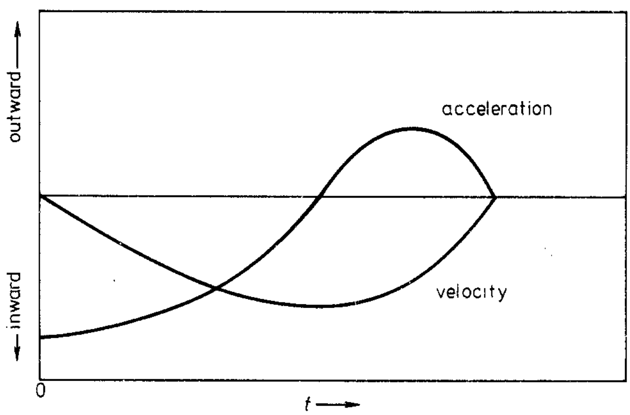

Figure 1. Plot of acceleration and velocity versus time (at fixed r).

would be (a) continued collapse beyond Newtonian densities to the relativistic regime, and (b) a single 'bounce' followed by unlimited expansion. These extreme cases will not be considered here.

In the shear-free examples above, $v = (\dot{\lambda}/\lambda)r$ and $a = (\ddot{\lambda}/\lambda)r$, so it is straightforward to follow the velocity and acceleration from $\dot{\lambda}$ and $\ddot{\lambda}$. We write down a $\lambda(t)$ corresponding to the case sketched in figure 1:

$$\lambda(t) = \lambda_0 \tau^{-4}(k\tau^4 - \tfrac{1}{2}\tau^2 t^2 + \tfrac{2}{3}\tau t^3 - \tfrac{1}{4}t^4), \tag{23}$$

where k, λ_0 and τ are positive constants.

$$\text{At } t = 0: \quad \lambda = k\lambda_0, \quad \dot{\lambda} = 0, \quad \ddot{\lambda} = -\lambda_0 \tau^{-2}, \quad \dddot{\lambda} = 4\lambda_0\tau^{-3}.$$

$$\text{At } t = \tau: \quad \lambda = (k - \tfrac{1}{12})\lambda_0, \quad \dot{\lambda} = \ddot{\lambda} = 0, \quad \dddot{\lambda} = -2\lambda_0\tau^{-3}.$$

It is instructive to follow the behaviour of the central pressure $p_c := p(0, t)$ for both examples 5.1 and 5.2 above.

From equations (16) and (22)

$$p_c = \alpha(\ddot{\lambda}/\lambda^2) + \beta/\lambda^4$$

where (α, β) are positive constants which take on different values corresponding to examples 5.1 and 5.2.

$$\dot{p}_c = \alpha[(\dddot{\lambda}/\lambda^2) - 2(\dot{\lambda}\ddot{\lambda}/\lambda^3)] - 4\beta\dot{\lambda}/\lambda^5.$$

At $t = 0$, $\dot{p}_c > 0$, and at $t = \tau$, $\dot{p}_c < 0$.

The central pressure increases monotonically to a maximum, and then falls to its equilibrium value. The time at which $(p_c)_{\max}$ is achieved depends on the particular model values (α, β) and is not simultaneous with $(\ddot{\lambda})_{\max}$.

7. Collapse with shear

7.1.

The functional form given in equation (7) is broken with the choice

$$m = \tfrac{4}{3}\pi m_0 \lambda^{-3} r^3 (1 + \lambda_0^{-2} r^2)^{-3/2}. \tag{24}$$

Equations (1)–(3) yield

$$\rho = m_0 \lambda^{-3}(1 + \lambda_0^{-2} r^2)^{-5/2}, \tag{25}$$

$$v = (\dot{\lambda}/\lambda) r (1 + \lambda_0^{-2} r^2), \tag{26}$$

$$p = m_0\{[2\dot{\lambda}^2 \lambda^{-5} r^2 + (\dot{\lambda}/\lambda)^{\cdot} \lambda^{-3}\lambda_0^2](1 + \lambda_0^{-2} r^2)^{-1/2} - 5\dot{\lambda}^2 \lambda^{-5}\lambda_0^2 (1 + \lambda_0^{-2} r^2)^{1/2} + \tfrac{2}{9}\pi G m_0 \lambda^{-6}\lambda_0^2 (1 + \lambda_0^{-2} r^2)^{-3}\} + h(t). \tag{27}$$

The boundary is obtained from equation (24) by demanding $m(r_b, t) = \text{constant}$†:

$$r_b = \lambda(1 - \lambda_0^{-2}\lambda^2)^{-1/2}, \tag{28}$$

† The condition $dr_b/dt = v(r_b, t)$ is automatically satisfied.

which yields $M = m(r_b, t) = \frac{4}{3}\pi m_0$. $h(t)$ in equation (27) is fixed by the condition $p(r_b, t) = 0$:

$$h(t) = -m_0[\dot{\lambda}^2\lambda^{-3}(2-5\lambda^{-2}\lambda_0^2)(1-\lambda_0^{-2}\lambda^2)^{-1/2} + (\dot{\lambda}/\lambda)^{\dot{}}\lambda^{-3}\lambda_0^2(1-\lambda_0^{-2}\lambda^2)^{1/2} + \tfrac{2}{9}\pi G m_0\lambda^{-6}\lambda_0^2(1-\lambda_0^{-2}\lambda^2)^3]. \tag{29}$$

The acceleration is given by

$$a = [(\ddot{\lambda}/\lambda) + (\dot{\lambda}/\lambda)^2 3\lambda_0^{-2}r^2]r(1+\lambda_0^{-2}r^2),$$

and so it is clear that the time profile given by equation (23) can be applied to this example of shearing collapse. The static end state is again the $n = 5$ polytrope.

7.2.

A different example which does not collapse to a static polytrope is given by

$$m = \tfrac{4}{3}\pi m_0 r_0^3\lambda^{-6}r^3(1+\lambda^{-1}r)^{-3}, \tag{30}$$

where $r_b = \lambda^2(r_0-\lambda)^{-1}$. Hence $\lambda(t) < r_0$ for this model. Equations (1)–(3) yield

$$\rho = m_0 r_0^3\lambda^{-6}(1+\lambda^{-1}r)^{-4}, \tag{31}$$

$$v = (\dot{\lambda}/\lambda)r(2+\lambda^{-1}r), \tag{32}$$

$$\begin{aligned} p = m_0 r_0^3\lambda^{-6}[&\tfrac{1}{3}(\ddot{\lambda}/\lambda)(1+\lambda^{-1}r)^{-3}(3r^2+6\lambda r+2\lambda^2) \\ &-(\dot{\lambda}/\lambda)^2(1+\lambda^{-1}r)^{-3}(\lambda^2+3\lambda r+2r^2) \\ &+(\dot{\lambda}/\lambda)^2(1+\lambda^{-1}r)^{-2}(2\lambda^2+6\lambda r+3r^2) - 2(\dot{\lambda}/\lambda)^2\lambda^2\ln(1+\lambda^{-1}r) \\ &+\tfrac{2}{45}\pi G m_0 r_0^3\lambda^{-4}(1+\lambda^{-1}r)^{-6}(1+6\lambda^{-1}r)] + h(t), \end{aligned} \tag{33}$$

where $h(t)$ is determined from $p(r_b, t) = 0$. The time profile given in equation (23) can also be applied to this example.

The equation of state in the static limit is given by

$$p = k_1\rho^{3/2} + k_2\rho^{5/4} - k_3.$$

8. Conclusion

It has been shown that by specifying the mass function, both shear-free and shearing collapse can be described. Two of the examples given had static end states corresponding to $n = 1$ and $n = 5$ Lane–Emden polytropes.

The models considered here represent the opposite end of a collapse model 'spectrum' from those models in which a single barotropic equation of state is fixed for the entire collapse. The freedom in specifying the time profile $\lambda(t)$ is allowed by having the systems collapse through a two-parameter family of equations of state. It is possible that some of the dynamically valid solutions will be excluded when the laws of thermodynamics are imposed upon assuming local thermodynamic equilibrium. This work is in progress.

For the purpose of investigating the stability of collapse against fragmentation, the exact models provided here, free of singularities in their entire domain, represent a reasonable starting point. Better models might be found if one could take a numerical collapse solution, fit an analytic function $m(r, t)$ to it, and then construct the remainder of the solution by the techniques of this paper.

Acknowledgments

I am grateful for the hospitality of the Physics Department, University of Victoria, where the bulk of this work was completed. The relativity group of Victoria provided a stimulating environment, and discussions with John Carminati and David Hobill were particularly valuable. I thank Bahram Mashhoon for the reference to Ritter's theorem, and for correcting a flaw in an early version of the generalised Ritter theorem. I had illuminating discussions with Paul Esposito during the early stages of this work. I thank the anonymous referees for useful comments.

References

Chandrasekhar S 1958 *Stellar Structure* (New York: Dover)
Cheng A 1978 *Astrophys. J.* **221** 320
Glass E N 1979 *J. Math. Phys.* **20** 1508
Hunter C 1967 in *Relativity Theory and Astrophysics* vol 2 *Galactic Structure* ed J Ehlers (Providence: American Mathematical Society)
—— 1977 *Astrophys. J.* **218** 834
Mansouri R 1977 *Ann. Inst. Henri Poincaré* **27** 175
McVittie G C 1956 *Astron. J.* **61** 451
Woodward P 1978 *Ann. Rev. Astron. Astrophys.* **16** 555